Hierarchical Modelling for non-Gaussian Spatial Data

Sudipto Banerjee¹ and Andrew O. Finley²

¹ Department of Forestry & Department of Geography, Michigan State University, Lansing Michigan, U.S.A.
² Biostatistics, School of Public Health, University of Minnesota, Minneapolis, Minnesota, U.S.A.

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Spatial Generalized Linear Models

- First stage: $Y(s_i)$ are conditionally independent given $\beta$ and $w(s_i)$, so $f(y(s_i)|\beta, w(s_i), \gamma)$ equals

$$h(y(s_i), \gamma) \exp \left( \gamma y(s_i) - \psi(y(s_i)) \right)$$

where $\phi(E(Y(s_i))) = h(s_i) = X^T(s_i)\beta + w(s_i)$ (canonical link function) and $\gamma$ is a dispersion parameter.

- Second stage: Model $w(s)$ as a Gaussian process:

$$w \sim N(0, \sigma^2 R(\phi))$$

- Third stage: Priors and hyperpriors.
- No process for $Y(s_i)$, only a valid joint distribution
- Not sensible to add a pure error term $\epsilon(s)$

Illustration

Binary spatial regression: forest/non-forest

We illustrate a non-Gaussian model for point-referenced spatial data:

- Objective is to make pixel-level prediction of forest/non-forest across the domain.
- Data: Observations are from 500 georeferenced USDA Forest Service Forest Inventory and Analysis (FIA) inventory plots within a 32 km radius circle in MN, USA.
- The response $Y(s)$ is a binary variable, with

$$Y(s) = \begin{cases} 
1 & \text{if inventory plot is forested} \\
0 & \text{if inventory plot is not forested}
\end{cases}$$

- Observed covariates include the coinciding pixel values for 3 dates of 30 × 30 m resolution Landsat imagery.

Illustration from:


- Often data sets preclude Gaussian modelling: $Y(s)$ may not even be continuous
- Example: $Y(s)$ is a binary or count variable
  - species presence or absence at location $s$
  - species abundance from count at location $s$
  - continuous forest variable is high or low at location $s$
- Replace Gaussian likelihood by exponential family member
  - Diggle Tawn and Moyeed (1998)

We are modelling with spatial random effects
- Introducing these in the transformed mean encourages means of spatial variables at proximate locations to be close to each other
- Marginal spatial dependence is induced between, say, $Y(s)$ and $Y(s')$, but observed $Y(s)$ and $Y(s')$ need not be close to each other
- Second stage spatial modelling is attractive for spatial explanation in the mean
- First stage spatial modelling more appropriate to encourage proximate observations to be close.

Comments

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Binary spatial regression: forest/non-forest

- We fit a generalized linear model where
  \[ Y(s_i) \sim \text{Bernoulli}(p(s_i)), \quad \text{logit}(p(s_i)) = x^T(s_i)\beta + w(s_i). \]

- Assume vague flat for \( \beta \), a Uniform\((3/32, 3/0.5)\) prior for \( \phi \), and an inverse-Gamma\((2, \cdot)\) prior for \( \sigma^2 \).

- Parameters updated with Metropolis algorithm using target log density:
  \[
  \ln \left( p(\Omega | Y) \right) \propto -\sigma_a + 1 + \frac{n}{2} \ln(\sigma^2) - \frac{\sigma_b}{\sigma^2} \ln(||R(\phi)||) - \frac{1}{2\sigma^2} w^T R(\phi)^{-1} w \\
  + \sum_{i=1}^n Y(s_i) \left( x^T(s_i)\beta + w(s_i) \right) + \frac{n}{2} \ln \left( 1 + \exp(x^T(s_i)\beta + w(s_i)) \right) \\
  + \ln(\sigma^2) + \ln(\phi - \phi_a) + \ln(\phi - \phi_b).
  \]

Covariates and proximity to observed FIA plot will contribute to increase precision of prediction.

### Posterior parameter estimates

Parameter estimates (posterior medians and upper and lower 2.5 percentiles):

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Estimates: 50% (2.5%, 97.5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept ((\theta_0))</td>
<td>82.39 (49.56, 120.46)</td>
</tr>
<tr>
<td>AprilTC1 ((\theta_1))</td>
<td>-0.27 (-0.45, -0.11)</td>
</tr>
<tr>
<td>AprilTC2 ((\theta_2))</td>
<td>-0.17 (0.07, 0.29)</td>
</tr>
<tr>
<td>AprilTC3 ((\theta_3))</td>
<td>-0.24 (-0.43, -0.08)</td>
</tr>
<tr>
<td>JulyTC1 ((\theta_4))</td>
<td>-0.04 (-0.25, 0.17)</td>
</tr>
<tr>
<td>JulyTC2 ((\theta_5))</td>
<td>0.09 (-0.01, 0.19)</td>
</tr>
<tr>
<td>JulyTC3 ((\theta_6))</td>
<td>0.01 (0.15, 0.16)</td>
</tr>
<tr>
<td>OctTC1 ((\theta_7))</td>
<td>-0.43 (-0.68, -0.22)</td>
</tr>
<tr>
<td>OctTC2 ((\theta_8))</td>
<td>-0.03 (0.19, 0.14)</td>
</tr>
<tr>
<td>OctTC3 ((\theta_9))</td>
<td>-0.26 (-0.46, -0.07)</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>1.358 (0.39, 2.42)</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.00182 (0.00065, 0.0032)</td>
</tr>
<tr>
<td></td>
<td>log(0.05)/( \phi ) (meters)</td>
</tr>
</tbody>
</table>

Classification of 15 20 \times 20 pixel areas (based on visual inspection of imagery) into non-forest (•), moderately forest (◦), and forest (no marker).