Dynamic models and SPDEs

Alan Gelfand$^1$ and Andrew O. Finley$^2$

$^1$ Department of Statistical Science, Duke University, Durham, North Carolina.
$^2$ Department of Forestry, Michigan State University, Lansing, Michigan.

September 9, 2014

For example:

Stage 1: Measurement equation

\[ y(s,t) = \mu(s,t) + \epsilon(s,t) ; \epsilon(s,t) \sim N(0,\sigma^2) . \]

\[ \mu(s,t) = x^T(s,t) \beta(s,t - 1) + w(s,t) \]

where \( w(s,t) \) a multivariate space-time process

Stage 2: Transition equation

\[ \beta_t = \beta_t - 1 + \eta_t, \eta_t \sim N(0,\Sigma) . \]

\[ \beta(s,t) = \beta(s,t - 1) + w(s,t) \]

where \( w(s,t) \) a multivariate space-time process

Dynamic models

- Dynamic models now a standard formulation for a wide variety of processes (also called Kalman filters, state space models and hidden Markov models)
- A first stage (or observational model), a second stage (or transition model), with third stage hyperparameters
- The first stage provides the data model while the second stage provides a latent dynamic process model
- The basic dynamic model takes the form:

\[ y_t = g(X_t, \theta_1) + \epsilon_t, \] observation equation with

\[ X_t = h(X_{t-1}, \theta_2) + \eta_t, \] transition equation.

Time is discrete with dynamics in the mean. Bayesian model fitting using the forward filter, backward sample (ffbs) algorithm

Connection to SPDE's

- Dynamic forms motivated by stochastic partial differential equations (SPDE's)
- Discretize time - resolution, dependence, features
- Many ecological diffusions: (i) emerging diseases such as avian flu or H1N1 flu; (ii) exotic organisms, i.e., invasive plants and animals; (iii) the evolution of the distribution of size or age of a species; (iv) the dynamics explaining phenomena such as transformation of landscape, deforestation, land use classifications, and urban growth.
- Objective: forecast likely spread in space and time with associated uncertainty. Evolution nonlinear and nonhomogeneous in space and time.
- For deterministic integro-differential equations or partial differential equations, how to add uncertainty?

An example

- Example: Data from the Breeding Bird Survey (BBS) to study the diffusion of the Eurasian collared dove.
- Let \( Z_{it} \) be the count in box \( i \) in year \( t \), let \( n_{it} \) be the number of visits to cell \( i \) in year \( t \) and let \( \lambda_{it} \) be the intensity for box \( i \) in year \( t \). Let \( Z_{it} \sim Po(n_{it}\lambda_{it}) \) with \( \log \lambda_{it} = w_{it} + \epsilon_{it}, \epsilon_{it} \) are i.i.d. (pure error).
- The \( w_{it} \) tell the diffusion story.
- More precisely, the dynamics here are associated with continuous space and discrete time, i.e., \( w_t(s) \).
- With \( t = (1, 2, ..., T) \) and locations \( s_1, s_2, ..., s_n \), let \( w_t = (w_t(s_1), w_t(s_2), ..., w_t(s_n))^T \).
- \[ |w_t| = |w_{t-1}| = |w_t| \] (first order Markov)
- Set \( w_t = Hw_{t-1} + \eta_t \) where \( \eta_t(s) \) incorporates spatial structure. We have a vector AR(1) model, \( H \) is the propagator matrix.

Again, \( w_t = Hw_{t-1} + \eta_t \). How to specify \( H \) ? \( H = I \) is nonstationary, explosive behavior with no interactions across space and time

\[ H = \text{Diag}(h) \] where \( \text{Diag}(h) \) has diagonal elements \( 0 < h_i < 1 \); not explosive but still no interactions.

In fact, what we seek is integro-difference equation (IDE) dynamics over the space of locations:

\[ w_t(s) = \int h(s, r; \phi)w_{t-1}(r)dr + \eta_t(s) \]

Here, \( h \) is a “redistribution kernel” that determines the rate of diffusion and the advection.

Alternatively, we could adopt \( v_t(s) = \int h(s, r; \phi)v_{t-1}(r)dr \),

\[ \log w_t(s) = \log v_t(s) + \eta_t(s) \]

Forms for \( h \): \( h(s, r; \phi), h(s, r; \phi(r)), h_t(s, r; \phi) \). Then, discretization of the spatial region will enable us to obtain \( H \)