Dynamic models and SPDEs

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Dynamic models

- Dynamic models now a standard formulation for a wide variety of processes (also called Kalman filters, state space models and hidden Markov models)
- A first stage (or observational model), a second stage (or transition model), with third stage hyperparameters
- The first stage provides the data model while the second stage provides a latent dynamic process model
- The basic dynamic model takes the form:

$$\mathbf{y}_t = g(\mathbf{X}_t, \boldsymbol{\theta}_1) + \boldsymbol{\epsilon}_t, \text{observation equation with}$$
$$\mathbf{X}_t = h(\mathbf{X}_{t-1}; \boldsymbol{\theta}_2) + \boldsymbol{\eta}_t, \text{transition equation.}$$

- Time is discrete with dynamics in the mean. Bayesian model fitting using the forward filter, backward sample (ffbs) algorithm

For example

- Stage 1: Measurement equation

$$y(s,t) = \mu(s,t) + \epsilon(s,t)\,;\ \epsilon(s,t) \overset{ind}{\sim} N\left(0, \sigma_\epsilon^2\right).$$

$$\mu(s,t) = \mathbf{x}^T(s,t)\,\tilde{\boldsymbol{\beta}}(s,t).$$

$$\tilde{\boldsymbol{\beta}}(s,t) = \boldsymbol{\beta}_t + \boldsymbol{\beta}(s,t)$$

Stage 2: Transition equation

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t,\ \boldsymbol{\eta}_t \overset{ind}{\sim} N_p\left(\mathbf{0}, \Sigma_{\boldsymbol{\eta}}\right).$$

$$\boldsymbol{\beta}(s,t) = \boldsymbol{\beta}(s,t-1) + \mathbf{w}(s,t)$$

where $\mathbf{w}(s,t)$ a multivariate space-time process

Connection to SPDE's

- Dynamic forms motivated by stochastic partial differential equations (SPDE's).
- Discretize time - resolution, dependence, features
- Many ecological diffusions: (i) emerging diseases such as avian flu or H1N1 flu; (ii) exotic organisms, i.e., invasive plants and animals; (iii) the evolution of the distribution of size or age of a species; (iv) the dynamics explaining phenomena such as transformation of landscape, deforestation, land use classifications, and urban growth.
- Objective: forecast likely spread in space and time with associated uncertainty. Evolution nonlinear and nonhomogeneous in space and time.
- For deterministic integro-differential equations or partial differential equations, how to add uncertainty?

An example

- Example: Data from the Breeding Bird Survey (BBS) to study the diffusion of the Eurasian collared dove.
- Let Z_{it} be the count in box i in year t, let n_{it} be the number of visits to cell i in year t and let λ_{it} be the *intensity* for box i in year t. Let $Z_{it} \sim$ Po$(n_{it}\lambda_{it})$ with log$\lambda_{it} = w_{it} + \epsilon_{it}$, ϵ_{it} are i.i.d. (pure error).
- The w_{it} tell the diffusion story.
- More precisely, the dynamics here are associated with continuous space and discrete time, i.e., $w_t(s)$.
- With $\mathbf{t} = (1, 2, ..., T)$ and locations $s_1, s_2, ..., s_n$, let $\mathbf{w}_t = (w_t(s_1), w_t(s_2), ..., w_t(s_n))^T$.
- $[\mathbf{w}_t|\mathbf{w}_0, \mathbf{w}_1, ...\mathbf{w}_{t-1}] = [\mathbf{w}_t|\mathbf{w}_{t-1}]$ (first order Markov)
- Set $\mathbf{w}_t = H\mathbf{w}_{t-1} + \boldsymbol{\eta}_t$ where $\eta_t(s)$ incorporates spatial structure. We have a vector AR(1) model, H is the *propagator* matrix.

cont.

- Again, $\mathbf{w}_t = H\mathbf{w}_{t-1} + \boldsymbol{\eta}_t$. How to specify H? $H = I$ is nonstationary, explosive behavior with no interactions across space and time.
- $H =$ Diag(h) where Diag(h) has diagonal elements $0 < h_i < 1$; not explosive but still no interactions.
- In fact, what we seek is integro-difference equation (IDE) dynamics over the space of locations: $w_t(s) = \int h(s, r; \phi) w_{t-1}(r) dr + \eta_t(s)$
- Here, h is a "redistribution kernel" that determines the rate of diffusion and the advection.
- Alternatively, we could adopt $v_t(s) = \int h(s, r; \phi) v_{t-1}(r) dr$, log$w_t(s) =$ log$v_t(s) + \eta_t(s)$
- Forms for h: $h(s, r; \phi)$, $h(s, r; \phi(r))$, $h_t(s, r; \phi)$. Then, discretization of the spatial region will enable us to obtain H