Multivariate spatial modeling

Cross-covariance functions satisfy certain properties:
\[ C_{XY}(s,t) = \text{cov}(X(s), Y(t)) \text{ for all } s, t \in D. \]

- Positive-definiteness for any finite collection of points:
  \[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j C_{X,Y}(s_i, t_j; \theta_Z) > 0 \text{ for all } a_i \in \mathbb{R}^2 \setminus \{0\}. \]

- Point-referenced spatial data often come as multivariate measurements at each location.
  - Examples:
    - Environmental monitoring: stations yield measurements on ozone, NO, CO, and PM$_{2.5}$.
    - Community ecology: assemblages of plant species due to water availability, temperature, and light requirements.
    - Forestry: measurements of stand characteristics age, total biomass, and average tree diameter.
    - Atmospheric modeling: at a given site we observe surface temperature, precipitation and wind speed

- We anticipate dependence between measurements
  - at a particular location
  - across locations

Bivariate Linear Spatial Regression

- A single covariate $X(s)$ and a univariate response $Y(s)$
- At any arbitrary point in the domain, we conceive a linear spatial relationship:
  \[ E[Y(s)|X(s)] = \beta_0 + \beta_1 X(s); \]
  where $X(s)$ and $Y(s)$ are spatial processes.

- Regression on uncountable sets:
  - Regress $\{Y(s) : s \in D\}$ on $\{X(s) : s \in D\}$.
  - Estimate $\beta_0$ and $\beta_1$.
  - Estimate spatial surface $\{X(s) : s \in D\}$.
  - Estimate spatial surface $\{Y(s) : s \in D\}$.

- Cross-covariance functions satisfy certain properties:
  \[ C_{XY}(s,t) = \text{cov}(X(s), Y(t)) = \text{cov}(Y(t), X(s)) = C_{YX}(t,s). \]

- Caution: $C_{XY}(s,t) \neq C_{XY}(t,s)$ and $C_{XY}(s,t) \neq C_{YX}(s,t)$.

- In matrix terms, $C_Z(s,t; \theta_Z) = C_Z(t,s; \theta_Z)$

- Positive-definiteness for any finite collection of points:
  \[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j C_Z(s_i, t_j; \theta_Z) > 0 \text{ for all } a_i \in \mathbb{R}^2 \setminus \{0\}. \]
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• Then, \( p(Y(s) \mid X(s)) = N(Y(s) \mid \beta_0 + \beta_1X(s), \sigma^2) \), where

\[
\begin{align*}
\beta_0 &= \mu_2 - T_{12}T_{11}^{-1} \beta_1, \\
\beta_1 &= T_{12}T_{11}^{-1}, \\
\sigma^2 &= T_{22} - T_{12}T_{11}^{-1}. 
\end{align*}
\]

• Regression coefficients are functions of process parameters.

• Estimate \( \{\mu_1, \mu_2, T_{11}, T_{12}, T_{22}\} \) by sampling from

\[ p(\phi) \times N(\mu, \delta, V_0) \times IW(T \mid r, S) \times N(Z \mid \mu, R(\phi) \otimes T) \]

immediately obtain posterior samples of \( \{\beta_0, \beta_1, \sigma^2\} \).

Bivariate Spatial Regression with Misalignment

• Rearrange the components of \( Z \) to

\[ Z = (X(s_1), X(s_2), \ldots, X(s_n), Y(s_1), Y(s_2), \ldots, Y(s_n))^\top \]
yields

\[
\begin{bmatrix} X \\ Y \end{bmatrix} \sim N \left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, T \otimes R(\phi) \right).
\]

• Priors: Wishart for \( T^{-1} \), normal (perhaps flat) for \( (\mu_1, \mu_2) \), discrete prior for \( \phi \) or perhaps a uniform on \((0, \pi)\max\ dist\).

• Estimation: Markov chain Monte Carlo (Gibbs, Metropolis, Slice, HMC/NUTS); Integrated Nested Laplace Approximation (INLA).

Hierarchical approach (contd.)

• \( X(s) \sim GP(\mu_X(s), C_X(\cdot; \theta_X)) \). Therefore,

\[ X \sim N(\mu_X, C_X(\theta_X)). \]

• \( C_X(\theta_X) \) is \( n \times n \) with entries \( C_X(s_i, s_j; \theta_X) \).

• \( e(s) \sim GP(0, C_e(\cdot; \theta_e)); C_e \) is analogous to \( C_X \).

\[ Y(s_i) = \beta_0 + \beta_1 X(s_i) + e(s_i), \quad \text{for} \quad i = 1, 2, \ldots, n. \]

• Joint distribution of \( Y \) and \( X \):

\[
\begin{bmatrix} X \\ Y \end{bmatrix} \sim N \left( \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} C_X(\theta_X) & \beta_1 C_X(\theta_X) \\ \beta_1 C_X(\theta_X) & \beta_1^2 C_X(\theta_X) \end{bmatrix} \right).
\]

where \( \mu_Y = \beta_01 + \beta_1 \mu_X \).

Coregionalization (Wackernagel)

• Separable models assume one spatial range for both \( X(s) \) and \( Y(s) \).

• Coregionalization helps to introduce a second “range parameter.”

• Introduce two “latent” independent GP’s, each having its own parameters:

\[ v_1(s) \sim GP(0, \rho_1(\cdot; \phi_1)) \quad \text{and} \quad v_2(s) \sim GP(0, \rho_2(\cdot; \phi_2)) \]

• Construct a bivariate process as the linear transformation:

\[
\begin{align*}
v_1(s) &= a_{11}v_1(s) + a_{12}v_2(s) \\
v_2(s) &= a_{21}v_1(s) + a_{22}v_2(s)
\end{align*}
\]
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Coregionalization

- Short form:
  \[ w(s) = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} = \mathbf{Av}(s) \]
- Cross-covariance of \( w(s) \):
  \[ C_w(s, t) = \begin{bmatrix} \rho_1(s, t, \phi_1) & 0 \\ 0 & \rho_2(s, t, \phi_2) \end{bmatrix} \]
- Cross-covariance of \( \mathbf{v}(s) \):
  \[ C_v(s, t) = \mathbf{A} C_v(s, t) \mathbf{A}^\top \].

It is a valid cross-covariance function (by construction).
- If \( s = t \), then \( C_w(s, s) = \mathbf{A} \mathbf{A}^\top \). No loss of generality to specify \( \mathbf{A} \) as (lower) triangular.

Generalizations

- Each location contains \( m \) spatial regressions
  \[ Y_k(s) = \mu_k(s) + w_k(s) + \epsilon_k(s), \quad k = 1, \ldots, m. \]
- Let \( v_k(s) \sim \text{GP}(\mu_k(s), \rho_k(s, s')) \), for \( k = 1, \ldots, m \) be \( m \) independent GP’s with unit variance.
- Assume \( \mathbf{w}(s) = \mathbf{A}(s) \mathbf{v}(s) \) arises as a space-varying linear transformation of \( \mathbf{v}(s) \). Then:
  \[ C_w(s, t) = \mathbf{A}(s) C_v(s, t) \mathbf{A}^\top(t) \]
  is a valid cross-covariance function.
- \( \mathbf{A}(s) \) is unknown!
  - Should we first model \( \mathbf{A}(s) \) to obtain \( C_w(s, s) \)?
  - Or should we model \( C_w(s, t) \) first and derive \( \mathbf{A}(s) \)?
  - \( \mathbf{A}(s) \) is completely determined from within-site associations.

Other approaches for cross-covariance models

- Convolutions of processes and covariance functions
- Latent dimension approach:
  - Apanasovich and Genton (Biometrika, 2010).
  - Apanasovich et al. (JASA, 2012).
- Multivariate Matérn family
  - Gneiting et al. (JASA, 2010).
- Nonstationary variants of coregionalization
  - Space-varying: Gelfand et al. (Test, 2010).
  - Dimension-reducing (over space): Guhaniyogi et al. (JABES, 2012).
  - Dimension-reducing (over outcomes): Ren and Banerjee (Biometrics, 2013).
  - Variogram modeling: De Iaco et al. (Math. Geo., 2003).