

Hierarchical Modelling for Spatiotemporal Data

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Specification:

- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

⇒ association in space, association in time

For point-referenced data, t continuous, Gaussian

$$Y(\mathbf{s}, t) = \mu(\mathbf{s}, t) + w(\mathbf{s}, t) + \epsilon(\mathbf{s}, t)$$

non-Gaussian data, $g(EY(\mathbf{s}, t) = \mu(\mathbf{s}, t) + w(\mathbf{s}, t))$

Don't treat time as a third coordinate (\mathbf{s}, t)

$$Cov(Y(\mathbf{s}, t), Y(\mathbf{s}', t')) = C(\mathbf{s} - \mathbf{s}', t - t')$$



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- **Nonseparable form:**

- Sum of independent separable processes
- Mixing of separable covariance functions
- Spectral domain approaches



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 - E , residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)

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- For $\epsilon_t(\mathbf{s})$, i.i.d. $N(0, \tau_t^2)$
- For $w_t(\mathbf{s})$
 - $w_t(\mathbf{s}) = \alpha_t + w(\mathbf{s})$
 - $w_t(\mathbf{s})$ independent for each t
 - $w_t(\mathbf{s}) = w_{t-1}(\mathbf{s}) + \eta_t(\mathbf{s})$, independent spatial process innovations

Dynamic spatiotemporal models

Measurement Equation

$$Y(\mathbf{s}, t) = \mu(\mathbf{s}, t) + \epsilon(\mathbf{s}, t); \quad \epsilon(\mathbf{s}, t) \stackrel{ind}{\sim} N(0, \sigma_\epsilon^2).$$

$$\mu(\mathbf{s}, t) = \mathbf{x}(\mathbf{s}, t)' \tilde{\beta}(\mathbf{s}, t).$$

$$\tilde{\beta}(\mathbf{s}, t) = \beta_t + \beta(\mathbf{s}, t)$$

Transition Equation

$$\beta_t = \beta_{t-1} + \eta_t, \quad \eta_t \stackrel{ind}{\sim} N_p(\mathbf{0}, \Sigma \eta)$$

$$\beta(\mathbf{s}, t) = \beta(\mathbf{s}, t-1) + \mathbf{w}(\mathbf{s}, t).$$