Introduction to Spatial Data and Models

Sudipto Banerjee\textsuperscript{1} and Andrew O. Finley\textsuperscript{2}

\textsuperscript{1} Biostatistics, School of Public Health, University of Minnesota, Minneapolis, Minnesota, U.S.A.

\textsuperscript{2} Department of Forestry & Department of Geography, Michigan State University, Lansing Michigan, U.S.A.

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Researchers in diverse areas such as climatology, ecology, environmental health, and real estate marketing are increasingly faced with the task of analyzing data that are:

- highly multivariate, with many important predictors and response variables,
- geographically referenced, and often presented as maps,
- and temporally correlated, as in longitudinal or other time series structures.

This motivates hierarchical modeling and data analysis for complex spatial (and spatiotemporal) data sets.
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⇒ motivates **hierarchical** modeling and data analysis for complex spatial (and spatiotemporal) data sets.
• **point-referenced data**, where $Y(s)$ is a random vector at a location $s \in \mathbb{R}^r$, where $s$ varies **continuously** over $D$, a fixed subset of $\mathbb{R}^r$ that contains an $r$-dimensional rectangle of positive volume;
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• **areal data**, where $D$ is again a fixed subset (of regular or irregular shape), but now partitioned into a **finite** number of areal units with well-defined boundaries;
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**areal data**, where $D$ is again a fixed subset (of regular or irregular shape), but now partitioned into a finite number of areal units with well-defined boundaries;

**point pattern data**, where now $D$ is itself random; its index set gives the locations of random events that are the spatial point pattern. $Y(s)$ itself can simply equal 1 for all $s \in D$ (indicating occurrence of the event), or possibly give some additional covariate information (producing a marked point pattern process).
First step in analyzing data
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First Law of Geography: Mean + Error
• First step in analyzing data

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• Mean: first-order behavior
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First Law of Geography: Mean + Error

Mean: first-order behavior

Error: second-order behavior (covariance function)

EDA tools examine both first and second order behavior

Preliminary displays: Simple locations to surface displays
First Law of Geography

\[ \text{data} = \text{mean} + \text{error} \]
Scallops Sites
Spatial surface observed at finite set of locations
\[ J = \{ s_1, s_2, \ldots, s_n \} \]

Tessellate the spatial domain (usually with data locations as vertices)

Fit an interpolating polynomial:
\[ f(s) = \sum_i w_i(J; s) f(s_i) \]

“Interpolate” by reading off \( f(s_0) \).

Issues:
- Sensitivity to tessellations
- Choices of multivariate interpolators
- Numerical error analysis
Introduction to spatial data and models

Scallops data: image and contour plots

Longitude
Latitude

-73.5 -73.0 -72.5 -72.0
39.0 39.5 40.0 40.5
Drop-line scatter plot
Surface plot

Scallops data: image and contour plots
Image contour plot
Locations form patterns
Surface features
Interesting plot arrangements
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**Statistical objectives:** Inference about the process \( Y(s) \); predict at new locations.
Suppose our spatial process has a mean, $\mu(s) = E(Y(s))$, and that the variance of $Y(s)$ exists for all $s \in D$. 

Strong stationarity: If for any given set of sites, and any displacement $h$, the distribution of $(Y(s_1), \ldots, Y(s_n))$ is the same as, $(Y(s_1 + h), \ldots, Y(s_n + h))$.

Weak stationarity: Constant mean $\mu(s) = \mu$, and $\text{Cov}(Y(s), Y(s+h)) = C(h)$: the covariance depends only upon the displacement (or separation) vector.

Strong stationarity implies weak stationarity.

The process is Gaussian if $Y = (Y(s_1), \ldots, Y(s_n))$ has a multivariate normal distribution.

For Gaussian processes, strong and weak stationarity are equivalent.
Suppose our spatial process has a mean, \( \mu(s) = E(Y(s)) \), and that the variance of \( Y(s) \) exists for all \( s \in D \).

- **Strong stationarity:** If for any given set of sites, and any displacement \( h \), the distribution of \((Y(s_1), \ldots, Y(s_n))\) is the same as, \((Y(s_1 + h), \ldots, Y(s_n + h))\).

- **Weak stationarity:** Constant mean \( \mu(s) = \mu \), and \( \text{Cov}(Y(s), Y(s + h)) = C(h) \): the covariance depends only upon the displacement (or separation) vector.
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Variograms

- Suppose we assume $E[Y(s + h) - Y(s)] = 0$ and define

$$E[Y(s + h) - Y(s)]^2 = Var (Y(s + h) - Y(s)) = 2\gamma(h).$$

This is sensible if the left hand side depends only upon $h$. Then we say the process is intrinsically stationary.
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Note that intrinsic stationarity defines only the first and second moments of the differences $Y(s + h) - Y(s)$. It says nothing about the joint distribution of a collection of variables $Y(s_1), \ldots, Y(s_n)$, and thus provides no likelihood.
Intrinsic Stationarity and Ergodicity

Relationship between $\gamma(h)$ and $C(h)$:

$$2\gamma(h) = Var(Y(s + h)) + Var(Y(s)) - 2Cov(Y(s + h), Y(s))$$

$$= C(0) + C(0) - 2C(h)$$

$$= 2[C(0) - C(h)].$$
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- Easy to recover $\gamma$ from $C$. The converse needs the additional assumption of ergodicity: $\lim_{\|u\| \to \infty} C(u) = 0$.

- So $\lim_{\|u\| \to \infty} \gamma(u) = C(0)$, and we can recover $C$ from $\gamma$ as long as this limit exists.

\[
C(h) = \lim_{\|u\| \to \infty} \gamma(u) - \gamma(h).
\]
When $\gamma(h)$ or $C(h)$ depends upon the separation vector only through the distance $\|h\|$, we say that the process is *isotropic*. In that case, we write $\gamma(\|h\|)$ or $C(\|h\|)$. Otherwise we say that the process is *anisotropic*.
When $\gamma(h)$ or $C(h)$ depends upon the separation vector only through the distance $||h||$, we say that the process is isotropic. In that case, we write $\gamma(||h||)$ or $C(||h||)$. Otherwise we say that the process is anisotropic.

If the process is intrinsically stationary and isotropic, it is also called homogeneous.
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If the process is intrinsically stationary and isotropic, it is also called homogeneous.

Isotropic processes are popular because of their simplicity, interpretability, and because a number of relatively simple parametric forms are available as candidates for $C$ (and $\gamma$). Denoting $\|h\|$ by $t$ for notational simplicity, the next two tables provide a few examples...
Some common isotropic variograms

<table>
<thead>
<tr>
<th>model</th>
<th>Variogram, $\gamma(t)$</th>
</tr>
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<tbody>
<tr>
<td>Linear</td>
<td>$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t &amp; \text{if } t &gt; 0 \ 0 &amp; \text{otherwise} \end{cases}$</td>
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Examples: Spherical Variogram

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- \( \lim_{t \to \infty} \gamma(t) = \tau^2 + \sigma^2 \); this asymptotic value of the semivariogram is called the **sill**. (The sill minus the nugget, \( \sigma^2 \) in this case, is called the **partial sill**.)
- Finally, the value \( t = 1/\phi \) at which \( \gamma(t) \) first reaches its ultimate level (the sill) is called the **range**, \( R \equiv 1/\phi \).
Examples: Spherical Variogram

b) spherical; $a_0 = 0.2$, $a_1 = 1$, $R = 1$
Some common isotropic covariograms

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Notes on exponential model

\[
C'(t) = \begin{cases} 
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\sigma^2 \exp(-\phi t) & \text{if } t > 0
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We define the **effective range**, \( t_0 \), as the distance at which this correlation has dropped to only 0.05. Setting \( \exp(-\phi t_0) \) equal to this value we obtain \( t_0 \approx \frac{3}{\phi} \), since \( \log(0.05) \approx -3 \).
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- Finally, the form of \( C(t) \) shows why the nugget \( \tau^2 \) is often viewed as a “nonspatial effect variance,” and the partial sill \( (\sigma^2) \) is viewed as a “spatial effect variance.”
The Matèrn Correlation Function

- Much of statistical modelling is carried out through correlation functions rather than variograms

\[ C(t) = \begin{cases} \sigma^2 & \text{if } t = 0 \\ \tau^2 + \sigma^2 & \text{if } t > 0 \end{cases} \]

\[ C(t) = \sigma^2 \left[ \frac{2^{\nu-1}}{\Gamma(\nu)} \right] K_\nu \left( 2 \sqrt{\nu} t \phi \right) \]

where \( K_\nu \) is the modified Bessel function of order \( \nu \) (computationally tractable)

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• Empirical Variogram:

$$\gamma(t) = \frac{1}{2|N(t)|} \sum_{\mathbf{s}_i, \mathbf{s}_j \in N(t)} (Y(\mathbf{s}_i) - Y(\mathbf{s}_j))^2$$

where $N(t)$ is the number of points such that $||\mathbf{s}_i - \mathbf{s}_j|| = t$ and $|N(t)|$ is the number of points in $N(t)$. 
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Grid up the \(t\) space into intervals \(I_1 = (0, t_1), I_2 = (t_1, t_2), \) and so forth, up to \(I_K = (t_{K-1}, t_K)\). Representing \(t\) values in each interval by its midpoint, we define:

\[
N(t_k) = \{(s_i, s_j) : |s_i - s_j| \in I_k\}, \quad k = 1, \ldots, K.
\]
Empirical variogram: scallops data
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