Hierarchical Modelling for Multivariate Spatial Data

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July 19, 2009
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We anticipate dependence between measurements

- at a particular location
- across locations
Each location contains $m$ spatial regressions

$$Y_k(s) = \mu_k(s) + w_k(s) + \epsilon_k(s), \; k = 1, \ldots, m.$$ 

- Mean: $\mu(s) = [\mu_k(s)]_{k=1}^m = [x_k^T(s)\beta_k]_{k=1}^m$
- Cov: $w(s) = [w_k(s)]_{k=1}^m \sim MVGP(0, \Gamma_w(\cdot, \cdot))$
  $$\Gamma_w(s, s') = [Cov(w_k(s), w_{k'}(s'))]_{k,k'=1}^m$$
- Error: $\epsilon(s) = [\epsilon_k(s)]_{k=1}^m \sim MVN(0, \Psi)$
  $$\Psi \text{ is an } m \times m \text{ p.d. matrix, e.g. usually } Diag(\tau_k^2)_{k=1}^m.$$
\( \mathbf{w}(\mathbf{s}) \sim MVGP(\mathbf{0}, \Gamma_{\mathbf{w}}(\cdot)) \) with

\[
\Gamma_{\mathbf{w}}(\mathbf{s}, \mathbf{s}') = [\text{Cov}(w_k(\mathbf{s}), w_{k'}(\mathbf{s}'))]_{k,k'=1}^m
\]

Example: with \( m = 2 \)

\[
\Gamma_{\mathbf{w}}(\mathbf{s}, \mathbf{s}') = \begin{pmatrix}
\text{Cov}(w_1(\mathbf{s}), w_1(\mathbf{s}')) & \text{Cov}(w_1(\mathbf{s}), w_2(\mathbf{s}')) \\
\text{Cov}(w_2(\mathbf{s}), w_1(\mathbf{s}')) & \text{Cov}(w_2(\mathbf{s}), w_2(\mathbf{s}'))
\end{pmatrix}
\]

For finite set of locations \( \mathcal{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\} \):

\[
\text{Var}\left(\left[\mathbf{w}(\mathbf{s}_i)\right]_{i=1}^n\right) = \Sigma_{\mathbf{w}} = \left[\Gamma_{\mathbf{w}}(\mathbf{s}_i, \mathbf{s}_j)\right]_{i,j=1}^n
\]
Properties:

- $\Gamma_w(s', s) = \Gamma_w^T(s, s')$
- $\lim_{s \to s'} \Gamma_w(s, s')$ is p.d. and $\Gamma_w(s, s) = \text{Var}(w(s))$.
- For sites in any finite collection $\mathcal{S} = \{s_1, \ldots, s_n\}$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_i^T \Gamma_w(s_i, s_j) u_j \geq 0 \text{ for all } u_i, u_j \in \mathbb{R}^m.$$ 

Any valid $\Gamma_w$ must satisfy the above conditions.

- The last property implies that $\Sigma_w$ is p.d.
- In complete generality:
  - $\Gamma_w(s, s')$ need not be symmetric.
  - $\Gamma_w(s, s')$ need not be p.d. for $s \neq s'$. 

Multivariate spatial modelling

Multivariate Gaussian process
Moving average or kernel convolution of a process:

- Let \( Z(\mathbf{s}) \sim GP(0, \rho(\cdot)) \). Use kernels to form:

\[
w_j(\mathbf{s}) = \int \kappa_j(\mathbf{u}) Z(\mathbf{s} + \mathbf{u}) d\mathbf{u} = \int \kappa_j(\mathbf{s} - \mathbf{s}') Z(\mathbf{s}') d\mathbf{s}'
\]

- \( \Gamma_w(\mathbf{s} - \mathbf{s}') \) has \((i, j)\)-th element:

\[
[\Gamma_w(\mathbf{s} - \mathbf{s}')]_{i,j} = \int \int \kappa_i(\mathbf{s} - \mathbf{s}' + \mathbf{u}) \kappa_j(\mathbf{u}') \rho(\mathbf{u} - \mathbf{u}') d\mathbf{u} d\mathbf{u}'
\]

Convolution of Covariance Functions:

- \( \rho_1, \rho_2, \ldots, \rho_m \) are valid covariance functions. Form:

\[
[\Gamma_w(\mathbf{s} - \mathbf{s}')]_{i,j} = \int \rho_i(\mathbf{s} - \mathbf{s}' - \mathbf{t}) \rho_j(\mathbf{t}) d\mathbf{t}.
\]
Constructive approach

- Let \( v_k(s) \sim GP(0, \rho_k(s, s')) \), for \( k = 1, \ldots, m \) be \( m \) independent GP’s with unit variance.
- Form the simple multivariate process \( v(s) = [v_k(s)]_{k=1}^m \):
  \[
  v(s) \sim MVGP(0, \Gamma_v(\cdot, \cdot))
  \]
  with \( \Gamma_v(s, s') = \text{Diag}(\rho_k(s, s'))_{k=1}^m \).
- Assume \( w(s) = A(s)v(s) \) arises as a space-varying linear transformation of \( v(s) \). Then:
  \[
  \Gamma_w(s, s') = A(s)\Gamma_v(s, s')A^T(s')
  \]
  is a valid cross-covariance function.
Constructive approach, contd.

- When \( s = s' \), \( \Gamma_v(s, s) = I_m \), so:
  \[
  \Gamma_w(s, s) = A(s)A^T(s)
  \]

- \( A(s) \) identifies with any square-root of \( \Gamma_w(s, s) \). Can be taken as lower-triangular (Cholesky).

- \( A(s) \) is unknown!
  - Should we first model \( A(s) \) to obtain \( \Gamma_w(s, s) \)?
  - Or should we model \( \Gamma_w(s, s') \) first and derive \( A(s) \)?
  - \( A(s) \) is completely determined from within-site associations.
Constructive approach, contd.

- If $A(s) = A$:
  - $w(s)$ is stationary when $v(s)$ is.
  - $\Gamma_w(s, s')$ is symmetric.
  - $\Gamma_v(s, s') = \rho(s, s')I_m \Rightarrow \Gamma_w = \rho(s, s')AA^T$

- Last specification is called intrinsic and leads to separable models:
  $$\Sigma_w = H(\phi) \otimes \Lambda; \ \Lambda = AA^T$$
Let \( y = [Y(s_i)]_{i=1}^{n} \) and \( w = [W(s_i)]_{i=1}^{n} \).

**First stage:**

\[
y | \beta, w, \Psi \sim \prod_{i=1}^{n} MVN (Y(s_i) | X(s_i)^T \beta + w(s_i), \Psi)
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**Second stage:**

$$w | \theta \sim MVN(0, \Sigma_w(\Phi))$$

where $\Sigma_w(\Phi) = [\Gamma_w(s_i, s_j; \Phi)]_{i,j=1}^n$. 

**Third stage:** Priors on $\Omega = (\beta, \Psi, \Phi)$.

Marginalized likelihood:

$$y | \beta, \theta, \Psi \sim MVN(X \beta, \Sigma_w(\Phi) + \mathbb{I} \otimes \Psi)$$
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\[
y | \beta, \theta, \Psi \sim \text{MVN} \left( X\beta, \Sigma_w(\Phi) + I \otimes \Psi \right)
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Choice: Fit as \([y|\Omega] \times [\Omega]\) or as \([y|\beta, w, \Psi] \times [w|\Phi] \times [\Omega]\).
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Conditional model:  
- Conjugate distributions are available for $\Psi$ and other variance parameters. Easy to program.
Choice: Fit as \( \mathbf{y} | \Omega \times [\Omega] \) or as \( \mathbf{y} | \beta, \mathbf{w}, \Psi \times [\mathbf{w} | \Phi] \times [\Omega] \).

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Marginalized model:
- need Metropolis or Slice sampling for most variance-covariance parameters. Harder to program.
- But reduced parameter space (no \( \mathbf{w} \)'s) results in faster convergence
- \( \Sigma_{\mathbf{w}}(\Phi) + I \otimes \Psi \) is more stable than \( \Sigma_{\mathbf{w}}(\Phi) \).
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- \(\Sigma_w(\Phi) + I \otimes \Psi\) is more stable than \(\Sigma_w(\Phi)\).

But what about \(\Sigma_w^{-1}(\Phi)\)?? Matrix inversion is EXPENSIVE \(O(n^3)\).
Recovering the w’s?

- Interest often lies in the spatial surface $w|y$. 

NOTE: With Gaussian likelihoods $[w|\Omega, y, X]$ is also Gaussian. With other likelihoods this may not be easy and often the conditional updating scheme is preferred.
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[w|y, X] = \int [w|\Omega, y, X] \times [\Omega|y, X] d\Omega
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- For each $\Omega^{(g)}$, draw $w^{(g)} \sim [w|\Omega^{(g)}, y, X]$
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Often we need to predict $Y(s)$ at a *new* set of locations \{$\tilde{s}_0, \ldots, \tilde{s}_m$\} with associated predictor matrix $\tilde{X}$.

Sample from predictive distribution:

$$[\tilde{y}|y, X, \tilde{X}] = \int [\tilde{y}, \Omega|y, X, \tilde{X}] d\Omega$$

$$= \int [\tilde{y}|y, \Omega, X, \tilde{X}] \times [\Omega|y, X] d\Omega,$$

$[\tilde{y}|y, \Omega, X, \tilde{X}]$ is multivariate normal. Sampling scheme:

- Obtain $\Omega^{(1)}, \ldots, \Omega^{(G)} \sim [\Omega|y, X]$
- For each $\Omega^{(g)}$, draw $\tilde{y}^{(g)} \sim [\tilde{y}|y, \Omega^{(g)}, X, \tilde{X}]$. 
Illustration from:

Slight digression – why we fit a model:

- Association between response and covariates, $\beta$, (e.g., ecological interpretation)

- Residual spatial and/or non-spatial associations and patterns (i.e., given covariates)

- Subsequent prediction
Study objectives:

- Evaluate methods for multi-source forest attribute mapping
- Find the “best” model, given the data
- Produce maps of biomass and uncertainty, by tree species
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Study area:
- USDA FS Bartlett Experimental Forest (BEF), NH
- 1,053 ha heavily forested
- Major tree species: American beech (BE), eastern hemlock (EH), red maple (RM), sugar maple (SM), and yellow birch (YB)
Bartlett Experimental Forest

Image provided by www.fs.fed.us/ne/durham/4155/bartlett
Response variables:

- Metric tons of total tree biomass per ha
- Measured on 437 \( \frac{1}{10} \) ha plots
- Models fit using random subset of 218 plots
- Prediction at remaining 219 plots
Covariates

- DEM derived elevation and slope
- Spring, Summer, Fall Landsat ETM+ Tasseled Cap features (brightness, greeness, wetness)
Candidate models

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5. $K$, different $\phi$, $Diag(\Psi)$
6. $K$, different $\phi$, $\Psi$
Model comparison

Deviance Information Criterion (DIC):

\[ D(\Omega) = -2 \log L(Data \mid \Omega) \]

\[ \overline{D(\Omega)} = E_{\Omega\mid Y}[D(\Omega)] \]

\[ p_D = \overline{D(\Omega)} - D(\bar{\Omega}); \bar{\Omega} = E_{\Omega\mid Y}[\Omega] \]

\[ DIC = \overline{D(\Omega)} + p_D. \]

Lower DIC is better.
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<td>8559</td>
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<tr>
<td>2</td>
<td>35</td>
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<td>34</td>
<td>8505</td>
</tr>
<tr>
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- Parameters: $K = 15$, $\phi = 5$, $Diag(\Psi) = 5$
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Focus on spatial cross-covariance matrix $K$ (for brevity).

Posterior inference of $\text{cor}(K)$, e.g., 50 (2.5, 97.5) percentiles:

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$E[\mathbf{w} \mid Data]$
\[ E[\mathbf{w} \mid \text{Data}] \]  

\[ E[\mathbf{w}^* \mid \text{Data}] \]
\[ E[Y^* \mid Data] \]
$$E[Y^* \mid Data]$$

$$P(2.5 < Y^* < 97.5 \mid Data)$$
Summary

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Extendable model template:
- Cluster plot sample design – multiresolution models
- Non-continuous response – general linear models
- Obs. over time and space – spatiotemporal models