Hierarchical Modeling for Spatio-temporal Data

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Specification:
- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

⇒ association in space, association in time
For point-referenced data, t continuous, Gaussian
\[ Y(s, t) = \mu(s, t) + w(s, t) + \epsilon(s, t) \]

non-Gaussian data, \[ g(EY(s, t)) = \mu(s, t) + w(s, t) \]

Don’t treat time as a third coordinate \((s, t)\)

\[ Cov(Y(s, t), Y(s', t')) = C(s - s', t - t') \]
Separable form:

\[ C(s - s', t - t') = \sigma^2 \rho_1(s - s'; \phi_1) \rho_2(t - t'; \phi_2) \]
Spatio-temporal Models

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Nonseparable form:

- Sum of independent separable processes
- Mixing of separable covariance functions
- Spectral domain approaches
Time discretized, $Y_t(s), t = 1, 2, \ldots T$
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• Type of data: time series or cross-sectional
Spatio-temporal Models

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- For time series data, exploratory analysis:
  
  - Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
  - Center by row averages of $Y$ yields $\bar{Y}$
  - Center by column averages of $Y$ yields $\bar{Y}_c$
  - Sample spatial covariance matrix: $\frac{1}{T} \bar{Y}_c \bar{Y}^T$
  - Sample autocorrelation matrix: $\frac{1}{n} \bar{Y}_c \bar{Y}_c^T$
  - $E$, residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)
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  - Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
  - Center by row averages of $Y$ yields $Y_{rows}$

- Sample spatial covariance matrix:
  $$\frac{1}{T-1} Y_{rows}^T Y_{rows}$$

- Sample autocorrelation matrix:
  $$\frac{1}{n-1} Y_{cols}^T Y_{cols}$$

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For \( \epsilon_t(s) \), i.i.d. \( N(0, \tau_t^2) \)

For \( w_t(s) \)
- \( w_t(s) = \alpha_t + w(s) \)
- \( w_t(s) \) independent for each \( t \)
- \( w_t(s) = w_{t-1}(s) + \eta_t(s) \), independent spatial process innovations
Dynamic spatiotemporal models

**Measurement Equation**

\[ Y(s, t) = \mu(s, t) + \epsilon(s, t); \quad \epsilon(s, t) \overset{ind}{\sim} N(0, \sigma^2_\epsilon). \]

\[ \mu(s, t) = x(s, t)'\tilde{\beta}(s, t). \]

\[ \tilde{\beta}(s, t) = \beta_t + \beta(s, t) \]

**Transition Equation**

\[ \beta_t = \beta_{t-1} + \eta_t, \quad \eta_t \overset{ind}{\sim} N_p(0, \Sigma\eta) \]

\[ \beta(s, t) = \beta(s, t - 1) + w(s, t). \]