Hierarchical Modeling for Spatio-temporal Data

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Specification:

- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

$\Rightarrow$ association in space, association in time
For point-referenced data, t continuous, Gaussian
$Y(\mathbf{s},t) = \mu(\mathbf{s},t) + w(\mathbf{s},t) + \epsilon(\mathbf{s},t)$

non-Gaussian data, $g(EY(\mathbf{s},t) = \mu(\mathbf{s},t) + w(\mathbf{s},t)$

Don't treat time as a third coordinate $(\mathbf{s},t)$

$$Cov(Y(\mathbf{s},t), Y(\mathbf{s}',t')) = C(\mathbf{s}-\mathbf{s}', t-t')$$

- Separable form:

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- Nonseparable form:
  - Sum of independent separable processes
  - Mixing of separable covariance functions
  - Spectral domain approaches

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  - E, residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)

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- For $\epsilon_t(\mathbf{s})$, i.i.d. $N(0, \tau_t^2)$
- For $w_t(\mathbf{s})$
  - $w_t(\mathbf{s}) = \alpha_t + w(\mathbf{s})$
  - $w_t(\mathbf{s})$ independent for each t
  - $w_t(\mathbf{s}) = w_{t-1}(\mathbf{s}) + \eta_t(\mathbf{s})$, independent spatial process innovations

Dynamic spatiotemporal models

Measurement Equation

$$Y(\mathbf{s},t) = \mu(\mathbf{s},t) + \epsilon(\mathbf{s},t); \quad \epsilon(\mathbf{s},t) \overset{ind}{\sim} N(0,\sigma_\epsilon^2).$$
$$\mu(\mathbf{s},t) = \mathbf{x}(\mathbf{s},t)'\tilde{\beta}(\mathbf{s},t).$$
$$\tilde{\beta}(\mathbf{s},t) = \boldsymbol{\beta}_t + \boldsymbol{\beta}(\mathbf{s},t)$$

Transition Equation

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \; \boldsymbol{\eta}_t \overset{ind}{\sim} N_p(\mathbf{0}, \Sigma_{\boldsymbol{\eta}})$$
$$\boldsymbol{\beta}(\mathbf{s},t) = \boldsymbol{\beta}(\mathbf{s},t-1) + \mathbf{w}(\mathbf{s},t).$$