Hierarchical Modelling for Univariate Spatial Data

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Univariate spatial models

Algorithmic Modelling

- Spatial surface observed at finite set of locations \( \mathcal{S} = \{s_1, s_2, \ldots, s_n\} \)
- Tessellate the spatial domain (usually with data locations as vertices)
- Fit an interpolating polynomial:
  \[
  f(s) = \sum_i w_i(\mathcal{S}; s_i) f(s_i)
  \]
- "Interpolate" by reading off \( f(s_0) \).
- Issues:
  - Sensitivity to tessellations
  - Choices of multivariate interpolators
  - Numerical error analysis

Univariate spatial models

Simple linear model

- Response: \( Y(s) \) at location \( s \)
- Mean: \( \mu = x^T(\mathbf{s})\beta \)
- Error: \( \epsilon(s) \overset{\text{iid}}{\sim} N(0, \tau^2) \)

Univariate spatial models

Simple linear model

- Assumptions regarding \( \epsilon(s) \):
  - \( \epsilon(s) \overset{\text{iid}}{\sim} N(0, \tau^2) \)
  - \( \epsilon(s_i) \) and \( \epsilon(s_j) \) are uncorrelated for all \( i \neq j \)
Univariate spatial models

**Spatial Gaussian processes (GP):**
- Say \( w(s) \sim GP(0, \sigma^2 R(\phi)) \) and
  \[
  \text{Cov}(w(s_1), w(s_2)) = \sigma^2 \rho(\phi; \|s_1 - s_2\|)
  \]
- Let \( w = [w(s_i)]_{i=1}^n \), then
  \[
  w \sim N(0, \sigma^2 R(\phi)), \quad \text{where } R(\phi) = [\rho(\phi; \|s_i - s_j\|)]_{i,j=1}^n
  \]

Realization of a Gaussian process:
- Changing \( \phi \) and holding \( \sigma^2 = 1 \):
  \[
  w \sim N(0, \sigma^2 R(\phi)), \quad \text{where } R(\phi) = [\rho(\phi; \|s_i - s_j\|)]_{i,j=1}^n
  \]
- Correlation model for \( R(\phi) \):
e.g., exponential decay
  \[
  \rho(\phi; t) = \exp(-\phi t) \quad \text{if } t > 0.
  \]
- Other valid models e.g., Gaussian, Spherical, Matérn.
- Effective range,
  \[
  t_0 = \ln(0.05)/\phi \approx 3/\phi
  \]

Univariate spatial regression

- First stage:
  \[
  y|\beta, w, \tau^2 \sim \prod_{i=1}^n N(Y(s_i) | X_i^T(s_i) \beta + w(s_i), \tau^2)
  \]
- Second stage:
  \[
  w|\sigma^2, \phi \sim N(0, \sigma^2 R(\phi))
  \]
- Third stage: Priors on \( \Omega = (\beta, \tau^2, \sigma^2, \phi) \)
- Marginalized likelihood:
  \[
  y|\Omega \sim N(X \beta, \sigma^2 R(\phi) + \tau^2 I)
  \]
- Note: Spatial process parametrizes \( \Sigma \):
  \[
  y = X \beta + \epsilon, \quad \epsilon \sim N(0, \Sigma), \quad \Sigma = \sigma^2 R(\phi) + \tau^2 I
  \]

Bayesian Computations
- Choice: Fit \( y|\Omega \times [\Omega] \) or \( y|\beta, w, \tau^2 \times [w|\sigma^2, \phi] \times [\Omega] \).
- Conditional model:
  - conjugate full conditionals for \( \sigma^2, \tau^2 \) and \( w \) – easier to program.
- Marginalized model:
  - need Metropolis or Slice sampling for \( \sigma^2, \tau^2 \) and \( \phi \). Harder to program.
  - But, reduced parameter space \Rightarrow faster convergence
  - \( \sigma^2 R(\phi) + \tau^2 I \) is more stable than \( \sigma^2 R(\phi) \).
- But what about \( R^{-1}(\phi) \) ?? EXPENSIVE!
Univariate spatial models

Where are the w’s?

- Interest often lies in the spatial surface \( w|y \).
- They are recovered from
  \[
  [w|y, X] = \int [w|\Omega, y, X] \times [\Omega|y, X] d\Omega
  \]
  using posterior samples:
  - Obtain \( \Omega^{(1)}, \ldots, \Omega^{(G)} \sim [\Omega|y, X] \).
  - For each \( \Omega^{(g)} \), draw \( w^{(g)} \sim [w|\Omega^{(g)}, y, X] \).

- NOTE: With Gaussian likelihoods \( [w|\Omega, y, X] \) is also Gaussian. With other likelihoods this may not be easy and often the conditional updating scheme is preferred.

Univariate spatial regression

Another look: \( [w(s)|y] \)

Residual plot: \( [w(s)|y] \)

Easting Northing

Another look: \( [w(s)|y] \)

Easting Northing

Often we need to predict \( Y(s) \) at a new set of locations \( \{s_0, \ldots, s_m\} \) with associated predictor matrix \( X \).

- Sample from predictive distribution:
  \[
  [\bar{y}|y, X, \bar{X}] = \int [\bar{y}|\Omega, y, X, \bar{X}] d\Omega
  = \int [\bar{y}|\Omega, y, X, \bar{X}] \times [\Omega|y, X] d\Omega,
  \]
  \( [\bar{y}|\Omega, y, X, \bar{X}] \) is multivariate normal. Sampling scheme:
  - Obtain \( \Omega^{(1)}, \ldots, \Omega^{(G)} \sim [\Omega|y, X] \).
  - For each \( \Omega^{(g)} \), draw \( \bar{y}^{(g)} \sim [\bar{y}|\Omega^{(g)}, y, X, \bar{X}] \).

Prediction: Summary of \( [Y(s)|y] \)
Modelling temperature: 507 locations in Colorado.

Simple spatial regression model:

\[ Y(s) = X(s)\beta + w(s) + \epsilon(s) \]

- \( w(s) \sim GP(0, \sigma^2 \rho(\cdot; \phi, \nu)) \);
- \( \epsilon(s) \overset{iid}{\sim} N(0, \tau^2) \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean (2.5%, 97.5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>2.827 (2.131, 3.866)</td>
</tr>
<tr>
<td>Elevation</td>
<td>-0.426 (-0.527, -0.333)</td>
</tr>
<tr>
<td>Precipitation</td>
<td>0.037 (0.002, 0.072)</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>0.134 (0.051, 1.245)</td>
</tr>
<tr>
<td>( \phi )</td>
<td>7.39E-3 (4.71E-3, 51.21E-3)</td>
</tr>
<tr>
<td>Range</td>
<td>278.2 (38.8, 476.3)</td>
</tr>
<tr>
<td>( \tau^2 )</td>
<td>0.051 (0.022, 0.092)</td>
</tr>
</tbody>
</table>

Temperature residual map

Elevation map

Residual map with elev. as covariate