Hierarchical Modelling for Spatialtemporal Data

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Specification:

- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

$\Rightarrow$ association in space, association in time

For point-referenced data, $t$ continuous, Gaussian

$$Y(s, t) = \mu(s, t) + w(s, t) + \epsilon(s, t)$$

non-Gaussian data, $g(EY(s, t)) = \mu(s, t) + w(s, t)$

Don’t treat time as a third coordinate $(s, t)$

$$\text{Cov}(Y(s, t), Y(s', t')) = C(s - s', t - t')$$
Spatio-temporal Models

Separable form:

\[ C(s - s', t - t') = \sigma^2 \rho_1(s - s'; \phi_1) \rho_2(t - t'; \phi_2) \]
Spatio-temporal Models

- **Separable form:**

\[
C(s - s', t - t') = \sigma^2 \rho_1(s - s'; \phi_1)\rho_2(t - t'; \phi_2)
\]

- **Nonseparable form:**
  - Sum of independent separable processes
  - Mixing of separable covariance functions
  - Spectral domain approaches
Time discretized, $Y_t(s), t = 1, 2, \ldots T$
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Type of data: time series or cross-sectional
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  • Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
Spatio-temporal Models

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For time series data, exploratory analysis:

- Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
- Center by row averages of $Y$ yields $Y_{rows}$

- Sample spatial covariance matrix:
- Sample autocorrelation matrix:
- $E$, residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)
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• For time series data, exploratory analysis:
  • Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
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  • Center by column averages of $Y$ yields $Y_{cols}$
• Time discretized, $Y_t(s), t = 1, 2, \ldots T$

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  • Arrange into an $n \times T$ matrix $Y$ with entries $Y_t(s_i)$
  • Center by row averages of $Y$ yields $Y_{rows}$
  • Center by column averages of $Y$ yields $Y_{cols}$
  • Sample spatial covariance matrix: $\frac{1}{T}Y_{rows}Y_{rows}^T$
Time discretized, $Y_t(s)$, $t = 1, 2, ... T$

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- Center by row averages of $Y$ yields $Y_{rows}$
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- sample spatial covariance matrix: $\frac{1}{T} Y_{rows} Y_{rows}^T$
- sample autocorrelation matrix: $\frac{1}{n} Y_{cols}^T Y_{cols}$
Time discretized, $Y_t(s), t = 1, 2, \ldots T$

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- $E$, residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)
Modeling: \( Y_t(s) = \mu_t(s) + w_t(s) + \epsilon_t(s) \), or perhaps
\[ g(E(Y_t(s))) = \mu_t(s) + w_t(s) \]
- **Modeling**: \( Y_t(s) = \mu_t(s) + w_t(s) + \epsilon_t(s) \), or perhaps
  \[ g(E(Y_t(s))) = \mu_t(s) + w_t(s) \]

- For \( \epsilon_t(s) \), i.i.d. \( N(0, \tau^2_t) \)
Spatio-temporal Models

Modeling: $Y_t(s) = \mu_t(s) + w_t(s) + \epsilon_t(s)$, or perhaps $g(E(Y_t(s))) = \mu_t(s) + w_t(s)$

For $\epsilon_t(s)$, i.i.d. $N(0, \tau_t^2)$

For $w_t(s)$

- $w_t(s) = \alpha_t + w(s)$
- $w_t(s)$ independent for each $t$
- $w_t(s) = w_{t-1}(s) + \eta_t(s)$, independent spatial process innovations
Dynamic spatiotemporal models

Measurement Equation

\[ Y(s, t) = \mu(s, t) + \epsilon(s, t); \quad \epsilon(s, t) \sim N(0, \sigma^2_\epsilon). \]
\[ \mu(s, t) = x(s, t)' \tilde{\beta}(s, t). \]
\[ \tilde{\beta}(s, t) = \beta_t + \beta(s, t) \]

Transition Equation

\[ \beta_t = \beta_{t-1} + \eta_t, \quad \eta_t \sim N_p(0, \Sigma \eta) \]
\[ \beta(s, t) = \beta(s, t-1) + w(s, t). \]