

# Hierarchical Modelling for Spatialtemporal Data

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## Specification:

- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

⇒ association in space, association in time

For point-referenced data,  $t$  continuous, Gaussian

$$Y(\mathbf{s}, t) = \mu(\mathbf{s}, t) + w(\mathbf{s}, t) + \epsilon(\mathbf{s}, t)$$

non-Gaussian data,  $g(EY(\mathbf{s}, t) = \mu(\mathbf{s}, t) + w(\mathbf{s}, t)$

**Don't** treat time as a third coordinate  $(\mathbf{s}, t)$

$$Cov(Y(\mathbf{s}, t), Y(\mathbf{s}', t')) = C(\mathbf{s} - \mathbf{s}', t - t')$$

- Separable form:

$$C(\mathbf{s} - \mathbf{s}', t - t') = \sigma^2 \rho_1(\mathbf{s} - \mathbf{s}'; \phi_1) \rho_2(t - t'; \phi_2)$$

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- Nonseparable form:

- Sum of independent separable processes
- Mixing of separable covariance functions
- Spectral domain approaches

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  - $E$ , residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)

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- For  $\epsilon_t(\mathbf{s})$ , i.i.d.  $N(0, \tau_t^2)$
- For  $w_t(\mathbf{s})$ 
  - $w_t(\mathbf{s}) = \alpha_t + w(\mathbf{s})$
  - $w_t(\mathbf{s})$  independent for each  $t$
  - $w_t(\mathbf{s}) = w_{t-1}(\mathbf{s}) + \eta_t(\mathbf{s})$ , independent spatial process innovations



## Dynamic spatiotemporal models

## Measurement Equation

$$Y(\mathbf{s}, t) = \mu(\mathbf{s}, t) + \epsilon(\mathbf{s}, t); \quad \epsilon(\mathbf{s}, t) \stackrel{ind}{\sim} N(0, \sigma_\epsilon^2).$$

$$\mu(\mathbf{s}, t) = \mathbf{x}(\mathbf{s}, t)' \tilde{\boldsymbol{\beta}}(\mathbf{s}, t).$$

$$\tilde{\boldsymbol{\beta}}(\mathbf{s}, t) = \boldsymbol{\beta}_t + \boldsymbol{\beta}(\mathbf{s}, t)$$

## Transition Equation

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{ind}{\sim} N_p(\mathbf{0}, \Sigma_{\boldsymbol{\eta}})$$

$$\boldsymbol{\beta}(\mathbf{s}, t) = \boldsymbol{\beta}(\mathbf{s}, t-1) + \mathbf{w}(\mathbf{s}, t).$$