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⇒ motivates *hierarchical* modeling and data analysis for complex spatial (and spatiotemporal) data sets.
• point-referenced data, where $y(s)$ is a random vector at a location $s \in \mathbb{R}^r$, where $s$ varies continuously over $D$, a fixed subset of $\mathbb{R}^r$ that contains an $r$-dimensional rectangle of positive volume;
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• **point pattern data**, where now \( D \) is itself random; its index set gives the locations of random events that are the spatial point pattern. \( y(s) \) itself can simply equal 1 for all \( s \in D \) (indicating occurrence of the event), or possibly give some additional covariate information (producing a marked point pattern process).
First step in analyzing data
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First Law of Geography: Mean + Error
• First step in analyzing data
• First Law of Geography: Mean + Error
• Mean: first-order behavior
Introduction to spatial data and models

Exploration of spatial data

- First step in analyzing data
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- Error: second-order behavior (covariance function)
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EDA tools examine both first and second order behavior
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• EDA tools examine both first and second order behavior

• Preliminary displays: Simple locations to surface displays
First Law of Geography

\[
\text{data} = \text{mean} + \text{error}
\]
Scallops Sites
Spatial surface observed at finite set of locations

\[ \mathcal{S} = \{ \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n \} \]

Tessellate the spatial domain (usually with data locations as vertices)

Fit an interpolating polynomial:

\[ f(\mathbf{s}) = \sum_i w_i(\mathcal{S}; \mathbf{s}) f(\mathbf{s}_i) \]

"Interpolate" by reading off \( f(\mathbf{s}_0) \).

Issues:

- Sensitivity to tessellations
- Choices of multivariate interpolators
- Numerical error analysis
Drop-line scatter plot
Surface plot
Image contour plot
Locations form patterns
Surface features

![3D surface plot of shrub density with Eastings and Northings axes, and Shrub Density on the y-axis.]
Interesting plot arrangements
Point-level modeling refers to modeling of spatial data collected at locations referenced by coordinates (e.g., lat-long, Easting-Northing).
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**Statistical objectives:** Inference about the process \( y(s) \); predict at new locations.
Suppose our spatial process has a mean, \( \mu(s) = E(y(s)) \), and that the variance of \( y(s) \) exists for all \( s \in D \).
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- **Strong stationarity:** If for any given set of sites, and any displacement \( h \), the distribution of \( (y(s_1), \ldots, y(s_n)) \) is the same as, \( (y(s_1 + h), \ldots, y(s_n + h)) \).

- **Weak stationarity:** Constant mean \( \mu(s) = \mu \), and \( \text{Cov}(y(s), y(s + h)) = C(h) \): the covariance depends only upon the displacement (or separation) vector.
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- For Gaussian processes, strong and weak stationarity are equivalent.
Suppose we assume $E[y(s + h) - y(s)] = 0$ and define

$$E[y(s + h) - y(s)]^2 = Var(y(s + h) - y(s)) = 2\gamma(h).$$

This is sensible if the left hand side depends only upon $h$. Then we say the process is intrinsically stationary.
Variograms

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Note that intrinsic stationarity defines only the first and second moments of the differences \( y(s + h) - y(s) \). It says nothing about the joint distribution of a collection of variables \( y(s_1), \ldots, y(s_n) \), and thus provides no likelihood.
Intrinsic Stationarity and Ergodicity

- Relationship between $\gamma(h)$ and $C(h)$:

$$2\gamma(h) = \text{Var}(y(s + h)) + \text{Var}(y(s)) - 2\text{Cov}(y(s + h), y(s))$$

$$= C(0) + C(0) - 2C(h)$$

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- So \( \lim_{\|u\| \to \infty} \gamma(u) = C(0) \), and we can recover \( C \) from \( \gamma \) as long as this limit exists.

\[
C(h) = \lim_{\|u\| \to \infty} \gamma(u) - \gamma(h).
\]
When \( \gamma(h) \) or \( C(h) \) depends upon the separation vector only through the distance \( ||h|| \), we say that the process is \textit{isotropic}. In that case, we write \( \gamma(||h||) \) or \( C(||h||) \). Otherwise we say that the process is \textit{anisotropic}. 
When $\gamma(h)$ or $C(h)$ depends upon the separation vector only through the distance $||h||$, we say that the process is isotropic. In that case, we write $\gamma(||h||)$ or $C(||h||)$. Otherwise we say that the process is anisotropic.

If the process is intrinsically stationary and isotropic, it is also called homogeneous.
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If the process is intrinsically stationary and isotropic, it is also called homogeneous.

Isotropic processes are popular because of their simplicity, interpretability, and because a number of relatively simple parametric forms are available as candidates for $C$ (and $\gamma$). Denoting $\|h\|$ by $t$ for notational simplicity, the next two tables provide a few examples...
Some common isotropic variograms

<table>
<thead>
<tr>
<th>model</th>
<th>Variogram, $\gamma(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t &amp; \text{if } t &gt; 0 \ 0 &amp; \text{otherwise} \end{cases}$</td>
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<tr>
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Examples: Spherical Variogram

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- Finally, the value \(t = 1/\phi\) at which \(\gamma(t)\) first reaches its ultimate level (the sill) is called the **range**, \(R \equiv 1/\phi\).
Examples: Spherical Variogram

b) spherical; \( a_0 = 0.2 \), \( a_1 = 1 \), \( R = 1 \)
Some common isotropic covariograms

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<tr>
<th>Model</th>
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Notes on exponential model

\[ C'(t) = \begin{cases} 
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Notes on exponential model

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- We define the effective range, \( t_0 \), as the distance at which this correlation has dropped to only 0.05. Setting \( \exp(-\phi t_0) \) equal to this value we obtain \( t_0 \approx 3/\phi \), since \( \log(0.05) \approx -3 \).
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- Finally, the form of \( C(t) \) shows why the nugget \( \tau^2 \) is often viewed as a “nonspatial effect variance,” and the partial sill \( (\sigma^2) \) is viewed as a “spatial effect variance.”
The Matèrn Correlation Function

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- The Matèrn is a very versatile family:

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C(t) = \begin{cases} 
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- \(\nu\) is a smoothness parameter (a fractal) controlling process smoothness
How do we select a variogram? Can the data really distinguish between variograms?

\[ \gamma(t) = \frac{1}{2N(t)} \sum_{s_i, s_j \in N(t)} (y(s_i) - y(s_j))^2 \]

where \( N(t) \) is the number of points such that \( \|s_i - s_j\| = t \) and \( |N(t)| \) is the number of points in \( N(t) \).

Grid up the \( t \) space into intervals \( I_1 = (0, t_1) \), \( I_2 = (t_1, t_2) \), and so forth, up to \( I_K = (t_{K-1}, t_K) \). Representing \( t \) values in each interval by its midpoint, we define:

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