Hierarchical Modeling for Univariate Spatial Data

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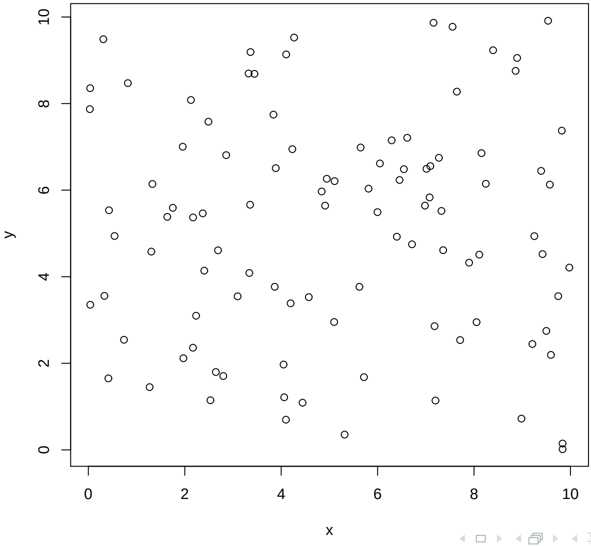
Spatial Domain



Algorithmic Modeling

- Spatial surface observed at finite set of locations $\mathscr{S} = \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_n\}$
- Tessellate the spatial domain (usually with data locations as vertices)
- Fit an interpolating polynomial:

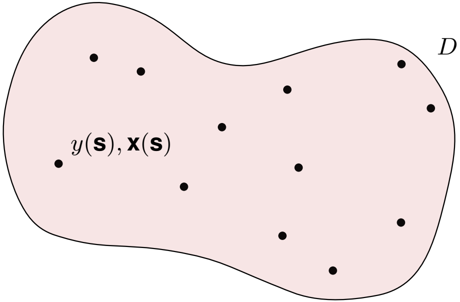
$$f(\mathbf{s}) = \sum_i w_i(\mathscr{S}; \mathbf{s}) f(\mathbf{s}_i)$$

- "Interpolate" by reading off $f(\mathbf{s}_0)$.
- Issues:
  - Sensitivity to tessellations
  - Choices of multivariate interpolators
  - Numerical error analysis

Simple linear model

$$y(\mathbf{s}) = \mu(\mathbf{s}) + \epsilon(\mathbf{s}),$$

- Response: $y(\mathbf{s})$ at location $\mathbf{s}$
- Mean: $\mu = \mathbf{x}(\mathbf{s})'\boldsymbol{\beta}$
- Error: $\epsilon(\mathbf{s}) \stackrel{iid}{\sim} N(0, \tau^2)$



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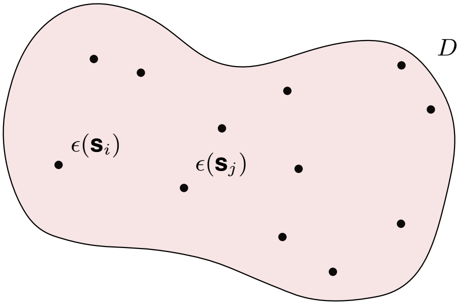

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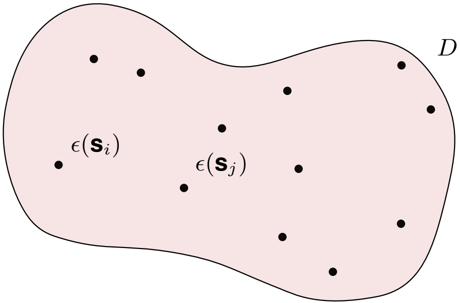
Assumptions regarding $\epsilon(\mathbf{s})$:

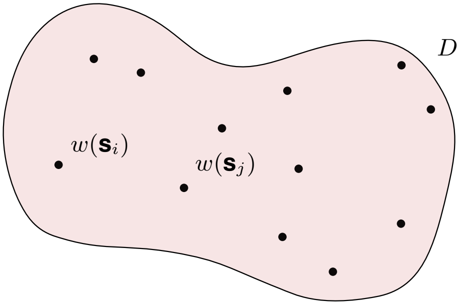
- $\epsilon(\mathbf{s}) \overset{iid}{\sim} N(0, \tau^2)$
- $\epsilon(\mathbf{s}_i)$ and $\epsilon(\mathbf{s}_j)$ are uncorrelated for all $i \neq j$



Spatial Gaussian processes (GP):

- Say $w(\mathbf{s}) \sim GP(0, \sigma^2 \rho(\cdot))$ and

$$Cov(w(\mathbf{s}_1), w(\mathbf{s}_2)) = \sigma^2 \rho\left(\phi; \|\mathbf{s}_1 - \mathbf{s}_2\|\right)$$



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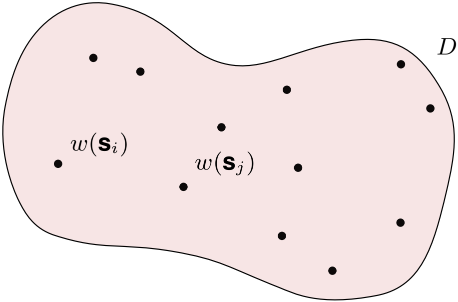
- Let $\mathbf{w} = [w(\mathbf{s}_i)]_{i=1}^n$, then

$$\mathbf{w} \sim N(\mathbf{0}, \sigma^2 R(\phi)), \text{ where } R(\phi) = [\rho(\phi; \|\mathbf{s}_i - \mathbf{s}_j\|)]_{i,j=1}^n$$



Realization of a Gaussian process:

- Changing ϕ and holding $\sigma^2 = 1$:

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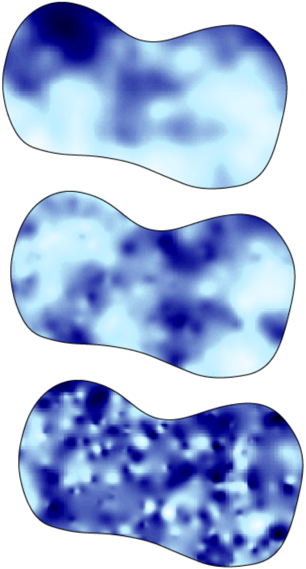

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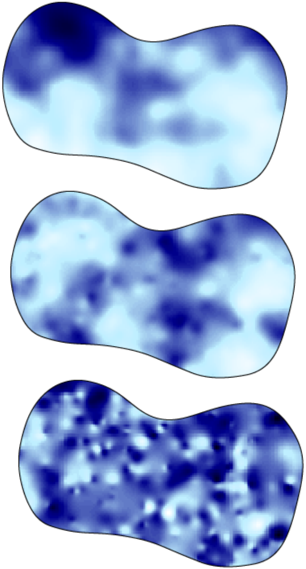
- Correlation model for $R(\phi)$:
  e.g., exponential decay

$$\rho(\phi; t) = \exp(-\phi t) \text{ if } t > 0.$$



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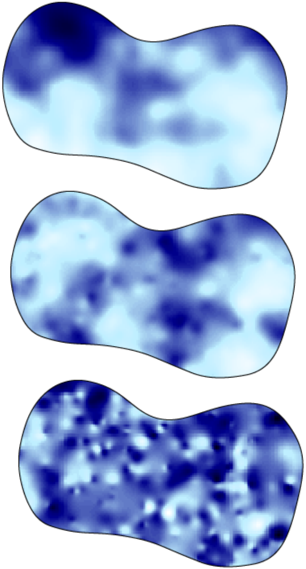
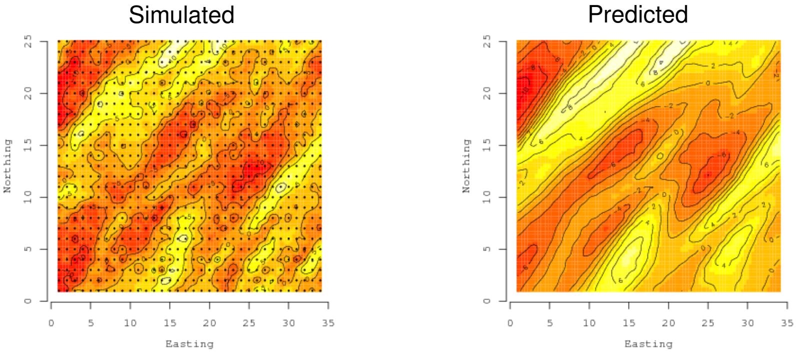
- Other valid models e.g., Gaussian, Spherical, Matérn.
- Effective range, $t_0 = -ln(0.05)/\phi \approx 3/\phi$



$\mathbf{w} \sim N(\mathbf{0}, \sigma_w^2 R(\phi))$ defines complex spatial dependence structures.

E.g., anisotropic Matérn correlation function:
$\rho(\mathbf{s}_i, \mathbf{s}_j; \phi) = \left(1/\Gamma(\nu)2^{\nu-1}\right)\left(2\sqrt{\nu d_{ij}})^{\nu}\kappa_{\nu}(2\sqrt{\nu d_{ij}}\right)$, where
$d_{ij} = (\mathbf{s}_i - \mathbf{s}_j)'\,\Sigma^{-1}\,(\mathbf{s}_i - \mathbf{s}_j)$, $\Sigma = G(\psi)\Lambda^2 G(\psi)'$. Thus, $\phi = (\nu, \psi, \Lambda)$.



Simple linear model + random spatial effects

$$y(\mathbf{s}) = \mu(\mathbf{s}) + w(\mathbf{s}) + \epsilon(\mathbf{s}),$$

- Response: $y(\mathbf{s})$ at some site
- Mean: $\mu = \mathbf{x}(\mathbf{s})'\beta$
- Spatial random effects: $w(\mathbf{s}) \sim GP(0, \sigma^2 \rho(\phi; \|\mathbf{s}_1 - \mathbf{s}_2\|))$
- Non-spatial variance: $\epsilon(\mathbf{s}) \overset{iid}{\sim} N(0, \tau^2)$

Hierarchical modeling

- First stage:

$$\mathbf{y}|\boldsymbol{\beta}, \mathbf{w}, \tau^2 \sim \prod_{i=1}^{n} N(y(\mathbf{s}_i)\,|\,\mathbf{x}(\mathbf{s}_i)'\boldsymbol{\beta} + w(\mathbf{s}_i), \tau^2)$$

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- Note: Spatial process parametrizes Σ:
  $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim N\left(\mathbf{0}, \Sigma\right),\ \Sigma = \sigma^2 R(\phi) + \tau^2 I$

Bayesian Computations

- Choice: Fit $[\mathbf{y}|\Omega] \times [\Omega]$ or $[\mathbf{y}|\boldsymbol{\beta}, \mathbf{w}, \tau^2] \times [\mathbf{w}|\sigma^2, \phi] \times [\Omega]$.

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- But what about $R^{-1}(\phi)$?? EXPENSIVE!

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  - For each $\Omega^{(g)}$, draw $\mathbf{w}^{(g)} \sim [\mathbf{w}|\Omega^{(g)}, \mathbf{y}, X]$
- NOTE: With Gaussian likelihoods $[\mathbf{w}|\Omega, \mathbf{y}, X]$ is also Gaussian. With other likelihoods this may not be easy and often the conditional updating scheme is preferred.

- Often we need to predict $y(\mathbf{s})$ at a *new* set of locations $\{\tilde{\mathbf{s}}_0, \ldots, \tilde{\mathbf{s}}_m\}$ with associated predictor matrix $\tilde{X}$.

- Sample from predictive distribution:

$$\begin{aligned}[\tilde{\mathbf{y}}|\mathbf{y}, X, \tilde{X}] &= \int [\tilde{\mathbf{y}}, \Omega|\mathbf{y}, X, \tilde{X}]d\Omega \\ &= \int [\tilde{\mathbf{y}}|\mathbf{y}, \Omega, X, \tilde{X}] \times [\Omega|\mathbf{y}, X]d\Omega,\end{aligned}$$

$[\tilde{\mathbf{y}}|\mathbf{y}, \Omega, X, \tilde{X}]$ is multivariate normal. Sampling scheme:

  - Obtain $\Omega^{(1)}, \ldots, \Omega^{(G)} \sim [\Omega|\mathbf{y}, X]$
  - For each $\Omega^{(g)}$, draw $\tilde{\mathbf{y}}^{(g)} \sim [\tilde{\mathbf{y}}|\mathbf{y}, \Omega^{(g)}, X, \tilde{X}]$.

Colorado data illustration

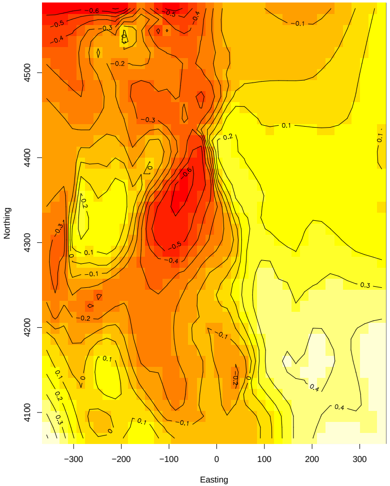
- Modeling temperature: 507 locations in Colorado.
- Simple spatial regression model:

$$y(\mathbf{s}) = \mathbf{x}(\mathbf{s})'\boldsymbol{\beta} + w(\mathbf{s}) + \epsilon(\mathbf{s})$$

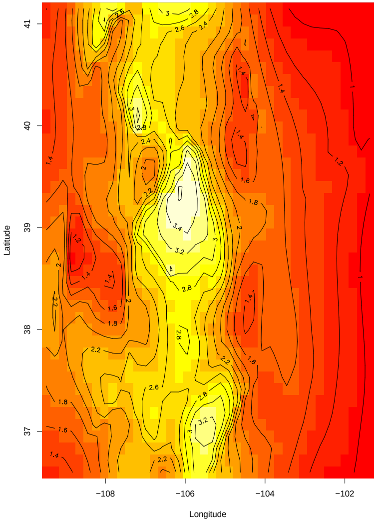
- $w(\mathbf{s}) \sim GP(0, \sigma^2\rho(\cdot;\phi,\nu))$; $\epsilon(\mathbf{s}) \stackrel{iid}{\sim} N(0, \tau^2)$

Parameters	50% (2.5%,97.5%)
Intercept	2.827 (2.131,3.866)
[Elevation]	-0.426 (-0.527,-0.333)
Precipitation	0.037 (0.002,0.072)
σ^2	0.134 (0.051, 1.245)
ϕ	7.39E-3 (4.71E-3, 51.21E-3)
Range	278.2 (38.8, 476.3)
τ^2	0.051 (0.022, 0.092)

Temperature residual map



Elevation map



Residual map with elev. as covariate